

Orbital Functional Geometry: A Synthesis

Overview of Papers I–XV

Preprint

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Abstract

This paper surveys the series *Orbital Functional Geometry* I–XV (2026), which develops a new geometric framework starting from a single equation.

The fundamental equation is:

$$e^{P(x)(az+b)} = f(x)$$

Its unique solution $z = \frac{1}{a} \left(\frac{\ln f}{P} - b \right)$ identifies $u = \ln f$ as the natural variable of the theory. Everything that follows is the unfolding of this observation.

From this equation emerges the Orbital Space $\mathcal{O}$ — a Hilbert space of positive analytic functions on S^1 with inner product $\langle f, g \rangle_{\mathcal{O}} = \frac{1}{2\pi} \int \ln f \cdot \ln g \, d\theta$, orbital Laplacian $\Delta_{\mathcal{O}} f = (\ln f)''$, and complete family of eigenfunctions $f_n^{(R,\phi)} = e^{R \cos(n\theta - \phi)}$ (von Mises functions).

The theory develops in three layers. *Geometry* (Papers I–VI): the orbital space, its spectral theory, two fundamental invariants Q and K , energy wells, compensation integral, and orbital curvature. *Analysis* (Papers VII–XII): the orbital zeta function $Z_{\mathcal{O}}(s) = \sum_k t_k^{-s}$, its double pole, relation to $\ln \xi$, full analytic continuation, tower of orbital zeta functions, Schrödinger linearisation, and complex extension with $\mathbb{Z}$ -graded gauge structure. *Applications* (Papers XIII–XV): Fisher–KPP equation, extension to $\mathbb{R}^n$, connection to Fisher information, and orbital geometry on Riemannian manifolds with Ricci curvature and Ricci flow.

The central thread is the substitution $u = \ln \psi$: it linearises the orbital Schrödinger equation on every Riemannian manifold, identifies the orbital potential with the Fisher information density, and gives the uncertainty principle via Cramér–Rao. This substitution was not imposed — it is the solution of the fundamental equation.

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1 The Fundamental Equation

Everything in this series originates from one equation:

$$e^{P(x)(az+b)} = f(x) \quad (1)$$

where $f : \Omega \rightarrow \mathbb{R}^+$ is a positive analytic function, P is a weight function, and a, b are real parameters.

The unique solution for z is:

$$z = \frac{1}{a} \left(\frac{\ln f(x)}{P(x)} - b \right)$$

This solution identifies $u = \ln f$ as the natural variable: the solution is linear in u . Everything that follows is the systematic development of this observation.

Remark 1 (Why this equation). *Equation (1) is the most general equation of the form “exponential of a linear function of z equals f ”. It encodes the relationship between a function f and its logarithm in the simplest non-trivial way. The geometry that emerges from it — the Orbital Space $\mathcal{O}$ — is therefore the natural geometry of the logarithm of analytic functions.*

2 Layer I: Geometry of the Orbital Space (Papers I–VI)

The space and its structure (I–II)

The *Orbital Space* $\mathcal{O}$ is the space of positive analytic functions on S^1 , equipped with:

$$\langle f, g \rangle_{\mathcal{O}} = \frac{1}{2\pi} \int_0^{2\pi} \ln f(\theta) \cdot \ln g(\theta) d\theta$$

$$\Delta_{\mathcal{O}} f = (\ln f)'', \quad \|f\|_{\mathcal{O}}^2 = \frac{1}{2\pi} \int_0^{2\pi} |\ln f|^2 d\theta$$

The discrete spectrum of $\Delta_{\mathcal{O}}$: eigenvalues $\lambda_n = n^2$, eigenfunctions $f_n^{(R,\phi)} = e^{R \cos(n\theta - \phi)}$ (von Mises functions), with quantisation condition $a^2 r^2 = 2/n^2$ and tower invariant $a_n r_n = \sqrt{2}$.

The Hilbert space $H_{\mathcal{O}}$ is complete. Every admissible f decomposes as $f = \prod_n f_n^{(R_n, \phi_n)}$.

Two fundamental invariants (III)

Two independent invariants parametrise $\mathcal{O}$:

$$Q = \frac{\ln f}{P} - b \quad (\text{geometric}), \quad K = b - \frac{2}{a^2} \quad (\text{spectral})$$

The (Q, K) plane carries three flows: b -flow (diagonal), a -flow (vertical), x_0 -flow (horizontal). Orbital energy: $E = Q + K$.

Energy wells and dynamics (IV–VI)

Applied to ζ , the natural parameters $a_\zeta = -\sqrt[3]{4}$, $b_\zeta = \sqrt[3]{4}/2$, $x_0 = 1/2$ satisfy $K = 0$ (spectral degeneracy) and centre the orbit on the critical line. Non-trivial zeros of ζ are energy wells: $E_\zeta \rightarrow -\infty$ at $s_k = 1/2 + it_k$.

The compensation integral (Paper V): $I(\delta) = \frac{\pi}{2\delta} \ln \frac{\delta}{2\pi}$, giving neutral point $\delta^* \approx 2\pi$.

Orbital curvature (Paper VI): $\kappa(T) = d^2/dT^2 \ln N(T) \sim -1/T^2$, with singularities at zeros of ζ : {singularities of κ } = $\{t_k\}$.

Triple representation of $\{t_k\}$: energy wells ($E_\zeta \rightarrow -\infty$), horizontal flow singularities ($\dot{Q} \rightarrow \infty$), curvature singularities (κ singular).

3 Layer II: Analysis of $Z_{\mathcal{O}}$ (Papers VII–XII)

The orbital zeta function (VII–X)

$$Z_{\mathcal{O}}(s) = \sum_{k=1}^{\infty} t_k^{-s}$$

converges for $\text{Re}(s) > 1$, double pole at $s = 1$:

$$Z_{\mathcal{O}}(s) \sim \frac{1}{2\pi(s-1)^2}$$

The double pole encodes orbital hyperbolicity: $N(T) \sim T \ln T / 2\pi$ (super-linear growth).

Generating relation (corrected in Paper VIII):

$$\ln \xi(s) = \ln \xi(0) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} Z_{\mathcal{O}}(2m)}{m} s^{2m}$$

$Z_{\mathcal{O}}$ generates ξ through its special values at even integers.

First special value (Paper VIII): $Z_{\mathcal{O}}(2) \approx B_\xi \approx 0.0231$ — the first special value coincides with the Hadamard constant.

Full analytic continuation (Paper IX): by iterated integration by parts, $Z_{\mathcal{O}}$ extends meromorphically to $\mathbb{C}$, with simple poles at $s = 0, -1, -2, \dots$ whose residues are moments of $S(T)$.

Recursive relation (Paper X): $Z_1(s) \approx -s \cdot Z_{\mathcal{O}}(s+1)$, connecting values in adjacent strips.

Tower of orbital zeta functions (Papers IX–X):

$$Z_{\mathcal{O}}^{(0)} = \zeta, \quad Z_{\mathcal{O}}^{(1)} = Z_{\mathcal{O}}, \quad Z_{\mathcal{O}}^{(n+1)}(s) = \sum_j \text{Im}(w_j^{(n)})^{-s}$$

Level n has pole of order $n+1$ at $s = 1$; the tower encodes n -point correlations of zeros. Under GUE statistics: self-similar tower.

Schrödinger linearisation and complex extension (XI–XII)

Linearisation theorem (Paper XI): $u = \ln \psi$ transforms the orbital Schrödinger equation into the free equation:

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} (\ln \psi)'' \psi \quad \xrightarrow{u=\ln \psi} \quad i\hbar \partial_t u = -\frac{\hbar^2}{2m} u''$$

Distinct from the LogSE of Rosen (1969) and Białyński-Birula–Mycielski (1976), which does not linearise.

Orbital potential (Paper XI): $V_{\mathcal{O}} = (\hbar^2/2m)(\partial_{\theta} \ln \psi)^2$ — positive definite, state-dependent.

Complex extension (Paper XII): $\mathcal{O}_{\mathbb{C}} = \bigsqcup_{k \in \mathbb{Z}} \mathcal{O}_k$ — $\mathbb{Z}$ -graded, $U(1)$ gauge structure, winding number $w(f) \in \mathbb{Z}$, instantons between sectors. Degeneracy reduces from ∞ (real) to 2 (complex): eigenfunctions $e^{e^{in\theta}}$, $n \in \mathbb{Z}$.

4 Layer III: Applications (Papers XIII–XV)

Fisher–KPP (Paper XIII)

Under $u = \ln \psi$: $\dot{u} = Du'' + Du'^2 + r(1 - e^u)$. The term $Du'^2 = V_{\mathcal{O}}$ is the orbital potential.

Under logit substitution $u = \text{logit}(\psi)$: generalised orbital potential $V_{\mathcal{O}}^{\text{KPP}} = D(1 - 2\sigma(u))u'^2$ vanishes at $\psi = 1/2$. **The propagation front is the zero of the orbital potential.** Linearisation at the front gives $c^* = 2\sqrt{Dr}$.

Extension to $\mathbb{R}^n$ and Fisher information (Paper XIV)

On $\mathbb{R}^n$: $\Delta_{\mathcal{O}} = \Delta(\ln \psi)$, linearisation holds, energy = H^1 norm of u .

Fisher identity: $V_{\mathcal{O}}^{(n)} = (\hbar^2/8m)|\nabla \ln p|^2$ and $\langle E \rangle = (\hbar^2/8m)I(p)$.

Orbital uncertainty principle: $\langle E \rangle \geq \hbar^2/8m(\Delta x)^2$ — same bound as Heisenberg, two independent routes.

Riemannian manifolds and Ricci flow (Paper XV)

On (M, g) : linearisation holds, eigenfunctions $\psi_k = e^{v_k}$ for v_k eigenfunction of Δ_g . On S^2 : $\psi_{lm} = e^{Y_{lm}}$.

Bochner identity: Ricci curvature controls $\Delta_g V_{\mathcal{O},g}$. $\text{Ric} > 0$ localises; $\text{Ric} < 0$ delocalises orbital states.

Geometric confinement: $\langle E \rangle_g \geq \hbar^2 C(M, g)/8md^2$.

Ricci flow: mean orbital potential decreases for $\text{Ric} > 0$; on Einstein metrics, orbital potential is eigenfunction of Δ_g .

5 The Central Thread

The variable $u = \ln \psi$ appears at every level:

Context	Role of $u = \ln f$	Paper
Fundamental equation	Solution $z \propto u$	I
Orbital invariant	$Q = u/P - b$	I
Hilbert structure	$\langle f, g \rangle = \int u_f u_g$	II
Schrödinger	Linearises orbital equation	XI
Orbital potential	$V_{\mathcal{O}} = (\hbar^2/2m) \nabla u ^2$	XI, XIV
Fisher information	$V_{\mathcal{O}} = (\hbar^2/8m) \nabla \ln p ^2$	XIV
Fisher–KPP	Reveals $V_{\mathcal{O}}$ in nonlinear PDE	XIII
Riemannian	Linearises on every (M, g)	XV

The substitution $u = \ln \psi$ was not chosen for convenience. It is the solution of the fundamental equation. The linearisation, the orbital potential, and the Fisher identity are not coincidences — they are consequences of working in the natural coordinates of $\mathcal{O}$.

6 What Is New

The following results do not appear in the literature to the author's knowledge:

The Orbital Space $\mathcal{O}$ as a Hilbert space of positive functions with inner product on logarithms.

The orbital Laplacian $\Delta_{\mathcal{O}} f = (\ln f)''$ and its spectral theory.

Von Mises functions as eigenfunctions of a natural geometric operator.

The tower $Z_{\mathcal{O}}^{(n)}$ of orbital zeta functions with poles of order $n + 1$.

The linearisation theorem: $u = \ln \psi$ gives the free Schrödinger equation, distinct from the LogSE which does not linearise.

The orbital potential as the Fisher information density: $V_{\mathcal{O}} \propto |\nabla \ln p|^2$.

The front of Fisher–KPP as the zero of the orbital potential.

Orbital geometry on Riemannian manifolds with Bochner–Ricci interaction and Ricci flow dynamics of the orbital potential.

The $\mathbb{Z}$ -graded complex orbital space with $U(1)$ gauge structure and instanton sectors.

7 Open Directions

1. **$\mathcal{O}$ on Calabi–Yau manifolds** (Ricci-flat): orbital dynamics without curvature contribution, governed by the Hessian term alone.
2. **Orbital index theorem:** an analogue of the Atiyah–Singer index theorem for the orbital Laplacian $\Delta_{\mathcal{O},g}$ on compact manifolds.
3. **Allen–Cahn in orbital coordinates:** application of $V_{\mathcal{O}}$ to $\partial_t \psi = D\psi'' + \psi(1 - \psi^2)$.
4. **Explicit first zero w_1 of $Z_{\mathcal{O}}$:** numerical evaluation of $\sum_k t_k^{-1/2-it} = 0$.
5. **Gross–Pitaevskii in $\mathcal{O}$:** orbital treatment of $i\partial_t \psi = -\psi'' + g|\psi|^2\psi$.
6. **Orbital geometry and mirror symmetry:** the complex extension $\mathcal{O}_{\mathbb{C}}$ and its $\mathbb{Z}$ -grading may connect to mirror symmetry on Calabi–Yau manifolds.
7. **The fundamental equation in higher dimension:** generalise $e^{P(x)(az+b)} = f(x)$ to $f : (M, g) \rightarrow \mathbb{R}^+$ with P a tensor field.

Acknowledgements

This series originated in 2026 from a question about the geometry of analytic functions. The equation $e^{P(x)(az+b)} = f(x)$ was not constructed to produce these results: they emerged from following the structure of the solution. Fifteen papers later, the theory extends from S^1 to arbitrary Riemannian manifolds, from spectral theory to Fisher information, from energy wells of ζ to the Ricci flow.

None of this was planned. All of it was found.

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